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A Criterion for Precompactness in the Space of Hypermeasures [1]

Abstract. Let Q denote the space of signed measures on the Borel σ-algebra of a separable complete space X. We endow Q with the norm $\|q\| = \sup |\int \varphi \mathrm{d}q|$, where the supremum is taken over all Lipschitz with constant 1 functions whose module does not exceed unity. This normed space is incomplete provided X is infinite and has at least one limit point. We call its completion the space of hypermeasures. Necessary and sufficient conditions for precompactness (=relative compactness) of a set of hypermeasures are found. They are similar to those of Prokhorov's and Fernique's theorems for measures.

Keywords: completion, hypermeasure, quasicontinuous functional, equiquasicontinuity, precompactness, tightness.

Let (X, ρ) be a metric space. We introduce the notation: $\mathcal{X}$ – the σ-algebra of Borel sets in X;

$$J(f) = \sup_{x,y \in X,\ x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)};$$

$\Phi \equiv \Phi(X, \rho) = \{\varphi \in \mathbb{R}^X :\ \forall\, x, y \in X\ |\varphi(x)| \leq 1\ \&\ |\varphi(x) - \varphi(y)| \leq \rho(x,y)\}$; $\mathrm{BL} \equiv \mathrm{BL}(X, \rho) = \bigcup\limits_{a>0} a\Phi$ – the class of all bounded Lipschitz functions on X; $\mathrm{Q} \equiv \mathrm{Q}(X)$ – the set of all charges (=signed measures) on $\mathcal{X}$; for $q \in \mathrm{Q}$ $qf = \int f \mathrm{d}q$ ($f \in \mathrm{L}_1(X, \mathcal{X}, |q|)$, the integration is performed over X), $\|q\|_\rho = \sup\limits_{\varphi \in \Phi(X,\rho)} |q\varphi|$.

The class BL contains all the functions $\varphi_{x_0}(x) = \rho(x_0, x)/(1 + \rho(x_0, x))$ and therefore separates the points of X. Then Lemma 1 [5] asserts that $\|\cdot\|_\rho$ is a norm in Q provided the space (X, ρ) is complete and separable. We denote the completion of Q w.r.t. this norm by $\overline{\mathrm{Q}}$ and call its elements *hypermeasures*. This definition does not imply that the space Q is incomplete

[1] This is the translation, with minor amendments, of the communication published in Ukrainian in *Dopovidi Natsionalnoï Akademii Nauk Ukrainy* [*Reports Nat. Acad. Sci. Ukr.*], 2006, No.9, p.38–41.

– but it will be so if the set X is infinite and has at least one limit point [1, p. 246]. (In [5] the last condition was missed.) Every hypermeasure can be realized as a linear functional on BL [5]. So we will say "hypermeasure on BL(X)". The value of a hypermeasure t on a function φ will be denoted, in the same manner as for charges, $t\varphi$. The convergence in $\overline{\mathrm{Q}}$ is that w.r.t. the norm.

The goal of this communication is to find the necessary and sufficient conditions for precompactness (=relative compactness) of an arbitrary set of hypermeasures. For measures, such conditions are provided by the classical Prokhorov's theorem [4] widely used in probability theory. A generalization of this theorem for Lusin spaces was proved by Fernique [2].

Theorem 1. *Let the space X be separable and complete. Then for any convergent sequence (t_n) of hypermeasures and any uniformly bounded pointwise converging to zero sequence (φ_n) of functions from* BL *such that* $\sup_n J(\varphi_n) < \infty$ *the relation $t_n\varphi_n \to 0$ holds.*

Proof. Consider first a stationary sequence of hypermeasures: $t_n = t \in \overline{\mathrm{Q}}$. Let (φ_n) be such a sequence of functions from BL that

$$M \equiv \sup_n \left(\sup_x |\varphi_n(x)| + J(\varphi_n) \right) < \infty$$

and $\varphi_n \to 0$. Assume that $t\varphi_n \not\to 0$. Then there exists a number $a > 0$ such that $|t\varphi_k| \geq 2a$ for infinitely many k. By the definition of hypermeasure there exists a charge q such that

$$\|t - q\| < \varepsilon. \tag{1}$$

By the dominated convergence theorem $q\varphi_n \to 0$, so $|q\varphi_k - t\varphi_k| > a$ for infinitely many k. On the other hand, $|q\varphi_k - t\varphi_k| \leq M\|q - t\|$, which together with the previous inequality yields $\|q - t\| > a/M$. This contradiction with (1) proves the theorem in the case $t_n = t$.

Let now the sequence (t_n) converge to some hypermeasure t. By the choice of (φ_n), $\varphi_n/M \in \Phi$ for all n. So $|(t_n - t)\varphi_n| \leq M\|t_n - t\| \to 0$. Now, the identity $t_n\varphi_n = (t_n - t)\varphi_n + t\varphi_n$ reduces the general case to that considered above. $\square$

Theorem 1 asserts the property of hypermeasures somewhat weaker (because of condition (1)) than the continuity property of charges. Let us formulate it in a more general form.

We say that a set T of linear functionals on $\mathrm{BL}(X)$ is *equiquasicontinuous* if for any $\varepsilon > 0$ there exists a Tykhonov's neighborhood of zero $U \in \mathbb{R}^X$ such that

$$\sup_{t\in T,\ \varphi\in U\cap\Phi} |t\varphi| < \varepsilon. \tag{2}$$

If herein T is a singleton: $T = \{t\}$, then the functional t will be called *quasicontinuous*. In this terminology, Theorem 1 asserts that every convergent sequence of hypermeasures is equiquasicontinuous, in particular, every hypermeasure is a quasicontinuous functional.

Lemma 1. *Let the space X be separable and complete. Then in order that a set T of hypermeasures on $\mathrm{BL}(X)$ be equiquasicontinuous it is necessary and sufficient that for any sequence $(t_n) \in T^{\mathbb{N}}$ and any pointwise converging to zero sequence $(\varphi_n) \in \Phi^{\mathbb{N}}$ the relation $t_n\varphi_n \to 0$ hold.*

Proof. `Sufficiency.` Separability of X and the definition of the class Φ imply existence of a decreasing sequence (U_n) of Tykhonov's neighborhoods of zero in $\mathbb{R}^X$ such that

$$\bigcap_n U_n \cap \Phi = \{0\}. \tag{3}$$

If T is not equiquasicontinuous, then one can find $\varepsilon > 0$ and, for each n, a hypermeasure $t_n \in T$ together with a function $\varphi_n \in U_n \cap \Phi$ such that

$$|t_n\varphi_n| \geq \varepsilon. \tag{4}$$

Then the sequence (φ_n) pointwise converges to zero, but $t_n\varphi_n \not\to 0$.

`Necessity.` Let a set T be equiquasicontinuous. Let us fix $\varepsilon > 0$ and choose a Tykhonov's neighborhood of zero $U \subset \mathbb{R}^X$ such that the inequality (2) holds. If a sequence $(\varphi_n) \in \Phi^{\mathbb{N}}$ pointwise converges to zero, then ultimately all its members belong to $U \cap \Phi$ and therefore for any sequence $(t_n) \in T^{\mathbb{N}}$ $|t_n\varphi_n| < \varepsilon$ if n is sufficiently large. □

The following main result is an analogue of the above-mentioned Fernique's theorem [2] and a generalization of Prokhorov's theorem where the tightness

[2] This analogy was noticed by the author when the Ukrainian original of the present communication had been already published. The paper [2] is not cited there.

condition is re-formulated in Fernique's form (equivalent to the classical one if X is Polish).

Theorem 2. *Let the space X be separable and complete. Then in order that a set $T \subset \overline{\mathrm{Q}}(X)$ be precompact it is necessary and sufficient that it be bounded and equiquasicontinuous.*

Proof. `Necessity.` Let us take an arbitrary sequence (U_n) of Tykhonov's neighborhoods of zero with property (3). If the set T is not equiquasicontinuous, then one can find $\varepsilon > 0$ such that any neighborhood U_n does not satisfy condition (2). And this means that for each n there exist $t_n \in T$ and $\varphi_n \in U_n \cap \Phi$ such that inequality (4) holds.

By the choice of (U_n) and condition (3) $\varphi_n(x) \to 0$ for any x. Since the set T is by assumption precompact, the sequence (t_n) contains a convergent subsequence $(t_n, n \in S \subset \mathbb{N})$. Then by theorem 1 $t_n\varphi_n \to 0$ as $n \to \infty, n \in S$, which contradicts to (4).

Necessity of boundedness is obvious.

`Sufficiency.` Let us take an arbitrary countable dense subset Ψ of the set Φ. Since by assumption $\sup_{t\in T} \|t\| < \infty$, any sequence $(t_n) \in T^{\mathbb{N}}$ contains a subsequence $(t_n, n \in S \subset \mathbb{N})$ such that for any $\psi \in \Psi$ the numeral sequence $(t_n\psi, n \in S)$ converges. Hence and from equiquasicontinuity of T, writing

$$|t_m\varphi - t_n\varphi| \le |t_m(\varphi - \psi)| + |t_m\psi - t_n\psi| + |t_n(\psi - \varphi)|,$$

we deduce fundamentality and therefore convergence of the sequence $(t_n\varphi, n \in S)$ for any $\varphi \in \mathrm{BL}$. Let us denote $t_0\varphi = \lim\limits_{n\to\infty,\ n\in S} t_n\varphi$ and show that for any sequence $(\eta_n, n \in S)$ of functions from Φ

$$t_n\eta_n - t_0\eta_n \to 0 \text{ as } n \to \infty, n \in S. \tag{5}$$

Separability of X and the definition of Φ imply that for an arbitrary infinite set $S_1 \subset S$ there exist an infinite set $S_2 \subset S_1$ and a function $\eta \in \Phi$ such that for any x $\eta_n(x) \to \eta(x)$ as $n \to \infty, n \in S_2$. Equuiquasicontinuity of T and the definition of t_0 entail quasicontinuity of the latter. Hence, writing

$$t_n\eta_n - t_0\eta_n = t_n(\eta_n - \eta) + t_n\eta - t_0\eta + t_0(\eta - \eta_n)$$

and recalling once again the definition of t_0, we get by Theorem 1 relation (5) with S_2 instead of S. Since the infinite set $S_1 \subset S$ which S_2 was extracted from

is arbitrary, it holds for $S_2 = S$, too. And this in view of arbitrariness of the sequence $(\eta_n) \in \Phi^S$ means that the sequence $(t_n, n \in S)$ is fundamental and therefore (recall that the space $\overline{\mathrm{Q}}$ is by construction complete) convergent.

So, any sequence in T contains a convergent subsequence, which means that T is precompact. □

Theorem 2 is applicable, in particular, to measures (here and below – finite). Let us show that in his case it turns to the above-mentioned Prokhorov's criterion. We say, somewhat extending the notion of tightness, that a set T of measures on $\mathcal{X}$ is *tight*, if for any $\varepsilon > 0$ there exists a completely bounded set $K \subset X$ such that for all $q \in T \quad q(X \setminus K) < \varepsilon$. If X is complete, then this definition is equivalent to the conventional one demanding compactness of K.

Lemma 2. *Let T be a bounded tight set of measures on the σ-algebra of Borel sets in a metric space X. Then T is equiquasicontinuous.*

Proof. It suffices to show that for any $\varepsilon > 0$ and completely bounded set $K \subset X$ there exists a Tykhonov's neighborhood of zero $U \subset \mathbb{R}^X$ such that for all $q \in T \;\; \varphi \in U \cap \Phi$ the inequality $\left|\int_K \varphi \mathrm{d}q\right| < \varepsilon$ holds.

By assumption there exists $C > 0$ such that $q(X) \leq C$ for all $q \in T$. So U will possess the required property if

$$\sup_{\varphi \in U \cap \Phi} \sup_{x \in K} |\varphi(x)| < \frac{\varepsilon}{C} \; .$$

Let us take an existing by the choice of K finite set $A \subset X$ such that each point $x \in K$ is less than $\varepsilon/2C$ apart from some point $a(x) \in A$. Then, by the definition of the class Φ, for any $\varphi \in \Phi$ and $x \in K \quad |\varphi(x) - \varphi(a(x))| < \varepsilon/2C$, so that one may put $U = \{f \in \mathbb{R}^X : \forall a \in A \; |f(a)| < \varepsilon/2C\}$. □

It is known [3], that the metric $\Lambda(q_1, q_2) \equiv ||q_1 - q_2||$ induces the $*$-weak convergence in the space of measures on $\mathcal{X}$ (in [3], the subspace of probability measures is considered, but the same argument applies to the whole space). Consequently, precompactness of a set T of measures on $\mathcal{X}$ is tantamount to the following property: for each sequence $(q_n) \in T^{\mathbb{N}}$ there exist a measure $q \;\; \mathcal{X}$ and an infinite set $S \subset \mathbb{N}$ such that for any $f \in \mathrm{C_b}(X) \quad q_n f \to q f$ as $n \to \infty, \; n \in S$. Then Prokhorov's theorem in the necessity part asserts that

under the condition of completeness and separability of X any precompact w.r.t. the norm of the space Q set of measures is tight. This together with Theorem 2 and Lemma 2 leads us to the following conclusion.

Corollary 1. *Let the space X be separable and complete. Then in order that a bounded set of measures on $\mathcal{X}$ be equiquasicontinuous it is necessary and sufficient that it be tight.*

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